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Antipodal covers of strongly regular graphs

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Abstract

Antipodal covers of strongly regular graphs which are not necessarily distance-regular are studied. The structure of short cycles in an antipodal cover is considered. In most cases, this provides a tool to determine if a strongly regular graph has an antipodal cover. In these cases, covers cannot be distance-regular except when they cover a complete bipartite graph. A relationship between antipodal covers of a graph and its line graph is investigated. Finally, antipodal covers of complete bipartite graphs and their line graphs are characterized in terms of weak resolvable transversal designs which are, in the case of maximal covering index, equivalent to affine planes with a parallel class deleted. This generalizes Drake's and Gardiner's characterization of distance-regular antipodal covers of complete bipartite graphs. Bipartite antipodal distance-regular graphs with odd diameter are characterized.

1. Introduction

First, the distance-regularity, a cover and an antipodal cover are defined. Then we explain why antipodal covers are interesting, and finally we give a summary of our main results.

A connected graph G is *distance-regular* if for any two vertices u and v, the number of vertices at a distance i from u and j from v depends only on i, j and the distance between u and v. For a detailed treatment and all the terms which we do not define see [3]. Distance-regular graphs of diameter two are called *strongly regular graphs.*

Suppose G is a graph with a partition π of its vertices into *cells* which are independent sets, and that between any two cells there are either no edges or there is a perfect matching. Let G/π be the graph with the cells of π as vertices and with two of them adjacent if and only if there is a matching between them. A graph with such

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a partition is called a *cover* of its quotient G/π and the cells are called *fibres*. If G/π is connected then all the fibres have the same size called *covering index* and usually denoted by r.

A graph G of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. Then 'being at distance d or zero' induces an equivalence relation on the vertices of G, and the equivalence classes are called *antipodal classes*. A cover of index r, in which the fibres are antipodal classes, is called *antipodal r-cover* of its quotient. If the partition π does not consist of all the singletons of the set, then the diameter of G is strictly larger than the diameter of G/π. For example, the cube is the unique distance-regular antipodal double-cover of the tetrahedron, (i.e., K_4), the line graph of the Petersen graph is the unique distance-regular antipodal triple-cover of K_5, the icosahedron is a distance-regular antipodal double-cover of K_6, and the dodecahedron is a distance-regular antipodal double-cover of the Petersen graph.

Distance-regular antipodal covers of complete bipartite graphs are equivalent to resolvable transversal designs which have already been studied extensively and many new covers have been found. Interesting connections with other combinatorial structures such as Hadamard matrices and projective planes were established. In 1989 Godsil suggested to extend these investigations to distance-regular antipodal covers of strongly regular graphs, see [9].

Perhaps the problem of looking for distance-regular antipodal covers of distance-regular graphs is easier when studied in a more general setting. For example, many questions about distance-transitive graphs can be answered more elegantly by techniques developed for distance-regular graphs (which contain all distance-transitive graphs). In the case when the diameter of the antipodal quotient is at least two there are only a few distance-regular antipodal covers known, unless of course they cover complete bipartite graphs. However, there are infinite families of antipodal covers (not necessary distance-regular) of nonbipartite strongly regular graphs, see [3, p. 386, Remark (i)].

Given a vertex x of a graph G we denote by $S_r(x)$ the set of vertices at a distance r from x. We call $S_r(x)$ the *sphere of radius r centered at x*. A graph induced by $S_i(x)$ is called the *ith subconstituent graph* corresponding to x. Recall that c_2 denotes the number of common neighbours of two vertices at distance two. Terwilliger (personal communications, 1993) has shown the following.

Theorem 1.1 (Terwilliger). *If an antipodal distance-regular graph H of diameter four is Q-polynomial, then each second subconstituent graph is either disconnected or of diameter four and it has six eigenvalues at most. In the second case the second subconstituent graphs are antipodal covers (not distance-regular unless c_2 is the same for all pairs of vertices at distance two).*

This result demonstrates that in some cases knowledge of antipodal covers (not necessarily distance-regular) could be used to construct distance-regular antipodal covers of diameter four. In addition, M. Brown (private communication, 1994) showed that subquadrangles of orthogonal generalized quadrangles are equivalent to certain antipodal double-covers (not necessarily distance-regular) of certain strongly regular graphs.

We will demonstrate that very often the condition for covers to be antipodal is restrictive enough to rule them out or to give their characterizations or some particular constructions. On the other hand, maybe there are too many strongly regular graphs and our problem of saying something about their (distance-regular) antipodal covers, in general, does not have a nice solution. For example, even to determine all distance-regular covers of complete graphs is still an open problem. Therefore, it would be good to identify which families of strongly regular graphs are of particular interest to us. For example, it is not known which generalized quadrangles have distance-regular antipodal covers [3, p. 279, Remark (ii)]. In this paper we will study antipodal covers of some large families of strongly regular graphs.

Let us introduce two such infinite families of strongly regular graphs which come from designs. The *line graph* (also called the *block graph*) of a design is the graph with lines (i.e., blocks) as vertices, and two of them being adjacent whenever there is a point incident to both lines. The line graph of a 2-$(v,s,1)$ design with $v-1>s(s-1)>0$ is strongly regular. As these designs are also called Steiner systems, their line graphs $S(s,v)$ are known as the *Steiner graphs*. The point graph of a Steiner system is a complete graph, thus the line graph of the Steiner system $S(2,v)$ is the line graph of the complete graph K_v, also called the *triangular graph* $T(v)$. The line graph of a transversal design $TD(s,v)$ is also strongly regular for $2\leqslant s\leqslant v$. For $s=2$ we get the *lattice graph* $K_v\times K_v$.

Strongly regular graphs with the same parameters as their complements are called *conference graphs*. They are the only strongly regular graphs that could have irrational eigenvalues. Since the multiplicities of eigenvalues are integral, the only distance-regular antipodal cover of a conference graph is the decagon. This has been first proved by Van Bon, see Brouwer et al. [3, p. 180]. Furthermore, it can be shown that the smallest eigenvalue of a strongly regular graph must be negative and cannot be -1. The strongly regular graph with the smallest eigenvalue $-m$, $m\geqslant 2$ integral, is with finitely many exceptions, either a complete multipartite graph, a Steiner graph, or the line graph of a transversal design, see Neumaier [11].

In Section 2, the structure of short cycles in an antipodal cover is investigated. It provides a tool to determine when the above two infinite families of strongly regular graphs allow antipodal covers. With one trivial exception, none of these covers is distance-regular. Analysis of this cycle structure also allows us to construct some antipodal covers. Under a mild restriction a bijective correspondence between antipodal covers of a graph and its line graph is established in Section 3. Antipodal covers of the lattice graphs and the complete bipartite graphs are characterized in Section 4. In the last section we characterize bipartite distance-regular antipodal graphs with odd diameter.

2. Antipodal covers, permutations and cycles

In this section an argument that implies new existence conditions for antipodal covers is given. We apply it to the complete multipartite graphs, the line graphs of transversal designs and the Steiner graphs.

It is sometimes convenient to record an r-cover of a graph by arbitrarily orienting its edges and then defining an *arc function* from the set of arcs to a set of permutations of order r. We can change orientation of any edge, if we replace the corresponding permutation with its inverse. We can choose this function to be the identity on a spanning tree. The following two results have been pointed out to me by Godsil (private communication).

Lemma 2.1 (Godsil). *Let G be a graph and H its antipodal cover of diameter D. Let C be a cycle in G with length less than D. Then the matchings between the fibres of H corresponding to the edges of C form a disjoint union of r cycles, all of the same length.*

Proof. The definition of an r-cover implies that the matchings corresponding to the edges of any path in G form in H a disjoint union of r paths of the same length as the path from G.

Now, suppose that there is an edge e between the beginning and the end of some path P of length less than $D-1$ in G. Then the matching in H corresponding to e matches the vertices of fibres of the beginning of P with the fibre of the end of P. This matching cannot extend the paths in H corresponding to P to paths, since they would begin and end in the same fibre and they would have length less than D. Hence these paths in H must be extended to the cycles we wanted to find. □

In other words, this lemma states that the product of the values of an arc function which determines H on the edges of any cycle of length less than D in G is the identity. Van Bon and Brouwer [1, 3, p. 144] have found necessary conditions for a distance-regular graph to have a distance-regular antipodal cover. They have dealt with the even and the odd diameter cases separately. They have shown that most of the classical distance-regular graphs have no distance-regular antipodal covers. We first used their results together with Godsil and Schade [9] to rule out almost all distance-regular antipodal covers of Steiner graphs and the line graphs of transversal designs. However, the above tool allows more general study and shorter proofs.

Corollary 2.2 (Godsil). *Let G be a graph of diameter d, girth g and with a spanning tree of diameter s. If H is an antipodal cover of diameter D, then $g \leqslant D \leqslant s+1$. In particular, G has no antipodal covers of diameter greater than $2d+1$.*

Gardiner [7,8] has shown that in an antipodal distance-regular graph the size of an antipodal class is bounded by the valency, and studied the cases when the bound is attained. Recall that a_1 denotes the number of triangles an edge lies in. For antipodal covers with diameter three we obtain the following result.

Lemma 2.3. *Let G be a distance-regular graph with an antipodal r-cover H of diameter three. Then $r \leqslant a_1(G)+1$. If the diameter of G is at least two, then also $r \leqslant c_2(G)$.*

Proof. Let u and v be adjacent vertices of H and let F be the fibre containing v. Since the diameter of H is three, the vertex u is at distance two from each of the $r-1$ vertices in $F\backslash\{v\}$. The middle vertices of the corresponding paths of length two between u and vertices of $F\backslash\{v\}$ induce distinct common neighbours of the two adjacent vertices of G, thus $r-1 \leqslant a_1(G)$. The proof of the second part is similar. $\square$

This lemma implies that the index of an antipodal cover with diameter three of the complete graph K_v is $v-1$ at most. When the covering index is $v-1$ the antipodal cover is distance-regular. This forecasts that antipodal covers with maximum covering index are interesting objects.

Proposition 2.4. *An antipodal cover of the line graph G of a transversal design $\mathrm{TD}(s,v)$, $s \leqslant v$, has diameter four when $s=2$, and diameter three otherwise.*

Proof. Let H be an antipodal r-cover of G determined by an arc function f on G. Suppose that $s=2$, i.e., G is the lattice graph $K_v \times K_v$, and that H has diameter three. Then $v>2$, and Lemma 2.3 implies $r=2=c_2(G)$. Moreover, (remember the proof of Lemma 2.3) this implies that each four-cycle in G has to lift to an eight-cycle in H. Hence there exists a colouring of the edges of G with red and blue colours (corresponding to the identity and the nonidentity permutations of f) such that each quadrangle contains an odd number of red edges. Then $K_3 \times K_3 \subseteq G$ is coloured this way as well. This graph has nine quadrangles and each edge lies in two of them (for, observe $C_3 \times C_3$ naturally embedded on the torus, i.e., $S^1 \times S^1$). If x is the number of quadrangles with three red edges, then $(3x+9-x)/2$ is the number of all the red edges. Contradiction!

Now, suppose that $s \geqslant 2$ and H has diameter five. Then by Lemma 2.1 the product of values of f on any triangle or quadrangle is the identity. Let us choose a spanning tree of diameter two in all the 'horizontal' and one 'vertical' copy of K_v. Then the edges of these spanning trees determine a spanning tree T of G. We choose an arc function f so that it is the identity on T, therefore it must be the identity on all the edges of those copies of K_v. Further, since each edge lies in a quadrangle or a triangle with at least three or two respectively edges in those copies of K_v, f must be the identity on all the edges.

Finally, let $s>2$, the diameter of H be at least four, and f be the identity on T. Then f is the identity on all the copies of K_v which contain edges of T, i.e., on all the horizontal and one vertical copy of K_v. Any skew edge of $E(G)$ (i.e., not horizontal

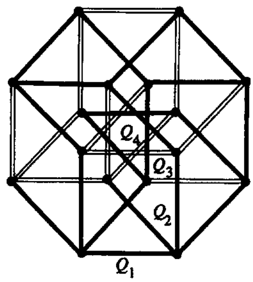

Fig. 1. A two-colouring of the four-cube.

or vertical) lies in a triangle with two edges in those copies of K_v, so f is the identity on all the skew edges. But now, for the same reason, f has to be the identity on all the vertical edges as well. □

It is easy to construct many antipodal double-covers with diameter three of the line graph G of a transversal design $\mathrm{TD}(s,v)$, $s>2$, We can accomplish this by assigning the identity permutation to the horizontal and the vertical edges of G and the non-identity permutation to all the other edges of G. A verification is left as an exercise.

Alternatively, if we switch the permutations on some small set of vertical edges from the same clique we can get another antipodal cover. For example, a switching on the edges of any matching of some vertical clique will do. Therefore, it seems that there are many nonisomorphic antipodal double-covers with diameter three and that there is no sense in classifying them. Note that a line graph of $\mathrm{TD}(s,v)$ is a conference graph when $v=2s-1$.

The two-colouring of edges mentioned in the first part of the above proof exists for the n-cube, $n>1$. We show how this can be used to obtain a very straightforward and elementary proof of the following result of Cohen and Tits [4,3, Proposition 9.2.10(i)], who used the fundamental group and the homology group of the n-cube to prove it.

Proposition 2.5 (Cohen and Tits). *There is the unique double-cover of the n-cube Q_n, $n>1$, having no quadrangles.*

Proof. Let us have $Q_1 \subset Q_2 \subset \cdots \subset Q_n$ (see Fig. 1) and let T be the spanning tree of Q_n with the following edges: $E(Q_1)$, the edges of Q_2 with one end in Q_1, the edges of Q_3 with one end in $Q_2, \ldots$, and the edges of Q_n with one end in Q_{n-1}. Let f be an arc function on Q_n which determines a double-cover having no quadrangles and is the identity on T. It suffices to prove that there exists a unique two-colouring of edges of Q_n (Fig. 1) (black for the identity permutation and white for the nonidentity permutation) such that each quadrangle contains an odd number of edges of each colour (i.e., each quadrangle lifts to an eight-cycle) and that the edges of $E(T)$ are black. Note that if we know the colours of three edges of some quadrangle, then we know the colour of the remaining edge as well. Q_2 has already three black edges, so the remaining edge is white. Suppose that we already know the colour of each edge in Q_i, $1<i<n$. Each edge of the set $E(Q_{i+1})\backslash(E(Q_i)\cup E(T))$ lies in a unique quadrangle with the remaining edges in $E(Q_i)\cup E(T)$, thus we know a colour of each edge in Q_{i+1}. Note that the two-colouring of $Q_{i+1}\backslash V(Q_i)$ is the opposite of the two-colouring of Q_i. By induction we know the whole two-colouring of Q_n. Since we have checked along the way that all quadrangles contain an odd number of edges of each colour, this is a unique double-cover of Q_n having no quadrangles. $\square$

Note that the first part of the proof of Proposition 2.4 implies that no other Hamming graph allows a double-cover, having no quadrangles.

Proposition 2.6. *Antipodal covers of Steiner graphs have diameter three.*

Proof. Let us choose a spanning tree T of a Steiner graph G which contains all the edges incident with some $\ell\in V(G)$ and no edges from $S_2(\ell)$. Let an arc function f, which determines an antipodal cover with diameter at least four, be the identity on the edges of this tree. By Lemma 2.1 f is the identity on all the edges with both ends in $S_1(\ell)$.

Now, let ℓ' be any element of $S_2(\ell)$ and $m\in S_1(\ell)\cap S_1(\ell')$. Denote the intersections of the line m with the lines ℓ and ℓ' by A and B, respectively. If $\ell'm$ is not an edge of T, then there is $\ell''\in S_1(\ell)$ such that $\ell'\ell''$ is an edge of T. Denote $\ell\cap\ell''$ and $\ell'\cap\ell''$ by C and D respectively. The line through A and D, denoted by m', is adjacent to ℓ,ℓ', m (or equal to m when $B=D$) and ℓ'' (or equal to ℓ'' when $A=C$). Since $m'\in S_1(\ell)$, we conclude that, in all these cases, f is the identity on edges $\ell'm'$ and $\ell'm$. This implies that f is the identity on all the edges between $S_1(\ell)$ and $S_2(\ell)$.

Finally, since every edge in $S_2(\ell)$ lies in a triangle with a vertex in $S_1(\ell)$, by Lemma 2.1, f is the identity on all the edges of G. $\square$

An infinite family of antipodal covers of the Steiner graphs will be constructed in the next section. Let us denote the complement of t cliques of order m by $K_{t(m)}$. A similar application of Lemma 2.1 as in Propositions 2.4 and 2.6 yields also the following result.

Proposition 2.7. *An antipodal cover of the complete multipartite graph $K_{t(m)}$, with $t,m\geqslant 2$, has diameter four for $t=2$ and diameter three otherwise.*

Antipodal double-covers with diameter three of $K_{t(m)}$ with $t>2$, can be constructed easily as well. For example, for natural labeling of the vertices of $K_{t(m)}$ with (i,j) for $i=1,\dots,t$ and $j=1,\dots,m$, choose f to be the identity on all the edges except $(i,j)(i',j)$ for $i\neq i'$, $j=1,\dots,m$, when $m>2$ and $(i,1)(i+1,2)$ for $i=1,\dots,t$ when $m=2$. A verification is left as an exercise.

The octagon is the only distance-regular antipodal cover of Hamming graphs, see Van Bon and Brouwer [1]. Therefore, it is also the only possible distance-regular

$$\begin{tikzcd} H \arrow[d] \arrow[r, phantom, "\xRightarrow{L}"] & L(H) \arrow[d] \\ G & L(G) \end{tikzcd}$$

Fig. 2. Antipodal covers and their line graphs.

antipodal cover of the lattice graphs. This also follows from Lemma 3.3 and the characterization of distance-regular line graphs in Brouwer et al. [3, p. 148] or, if you prefer, from the fact that multiplicities of eigenvalues are integral.

Gardiner [8] has proved that the diameter of distance-regular antipodal covers of a graph with diameter d can only be $2d$ or $2d+1$. Hence the graphs with diameter two can only have distance-regular antipodal covers of diameter four or five. Results in this section are particularly interesting because they yield the following conclusion: only finitely many strongly regular graphs with the smallest integral eigenvalue $-m$, $m \geqslant 2$, can have distance-regular antipodal covers.

3. Antipodal covers and line graphs

Lemma 2.1 implies that in the case of antipodal covers of diameter at least four the cliques of its antipodal quotient 'lift' to cliques (see Lemma 3.3). This suggests a study of antipodal covers of line graphs (see Fig. 2). Another reason for this is that the lattice graphs (i.e., the line graphs of the complete bipartite graphs) allow antipodal covers of diameter four. For a graph G and a subset S of its edges we define the *closure* of S to be the subset of $E(G)$ obtained from S by recursively adding edges, which are outside the current set and which form a triangle with edges of the current set, until no such edges remain. This closure is closely related to that defined in Bondy and Murty [2, p. 56], which was used to study hamiltonicity of graphs, and whose proof that it is well defined is very similar. The reader is encouraged to use the above closure operation to show that Lemma 2.1 cannot be used to rule out antipodal covers with diameter four of the lattice graphs $K_v \times K_v$.

A graph is a line graph if and only if its edges can be partitioned into cliques in such a way that no vertex lies in more than two of the cliques, see Harary [10, p. 74]. We shall usually identify the vertices of a graph with the cliques of its line graph.

Theorem 3.1. *Let H be an antipodal r-cover with diameter $D>1$ of a graph G. Then $L(H)$ is an r-cover with diameter D of $L(G)$. If $S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ for any two adjacent vertices u and v of H, then $L(H)$ is an antipodal cover of $L(G)$.*

Proof. Evidently, the line graph $L(H)$ is an r-cover of $L(G)$. Since H is antipodal, the distance between any two vertices of $L(H)$ is D at most, and the vertices in the same fibre of $L(H)$ are at distance D. It remains to show that for any two edges $e_1 = uv$ and $e_2 = tw$ which are not in the same fibre, there is a path in H of length $D-2$ at most between a vertex incident with e_1 and a vertex incident with e_2. We can assume that the distances from w and t to u are at least $D-1$. Since H is antipodal, we can assume that $w \in S_{D-1}(u)$. The condition $S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ is equivalent to $S_{D-1}(u) \subseteq S_D(v) \cup S_{D-2}(v)$, so we can assume that $w \in S_D(v)$. Then $t \in S_{D-1}(u)$, since otherwise e_1 and e_2 would be in the same fibre. Finally, as H is antipodal, $t \in S_{D-2}(v)$, which implies the existence of a desired path. □

Recall that in a distance-regular graph $a_i = |S_i(u) \cap S_1(v)|$ for any two vertices u and v at distance i. If H is either distance-regular with $a_{D-1} = 0$ or bipartite, then $S_{D-1}(u) \cap S_{D-1}(v) = \emptyset$ for any two adjacent vertices u and v of H.

Corollary 3.2. *Let H be a distance-regular antipodal r-cover with diameter D of a graph G. Then $L(H)$ is an r-cover with diameter D of $L(G)$. The line graph $L(H)$ is an antipodal r-cover of $L(G)$ if and only if $a_{D-1}(H) = 0$.*

Proof. If $a_{D-1}(H) \neq 0$, then for two adjacent vertices u and v of H there exists $w \in S_{D-1}(u) \cap S_{D-1}(v)$. If t is a neighbour of w in $S_D(u)$, then the edges uv and tw are at distance D and they are not from the same fibre. Therefore, the fibres of $L(H)$ are not antipodal classes. The converse follows directly from Theorem 3.1. □

For $D \geqslant 4$ or for $D = 3$ and $r = 2$, the condition $a_{D-1}(H) = 0$ translates to 'H being a triangle free'. Most of the known infinite families of feasible intersection arrays of distance-regular antipodal covers of strongly regular graphs are triangle free, see Brouwer et al. [3, pp. 417–425]. The complete bipartite graph $K_{m,m}$ with a perfect matching deleted is a triangle free distance-regular antipodal double-cover of K_m, so, by Corollary 3.2, its line graph is an antipodal double-cover of $L(K_m)$, i.e., the Steiner system $S(2, m)$. The following statement is an obvious consequence of Lemma 2.1 and the characterization of a line graph mentioned above.

Lemma 3.3. *Let G be a graph and A an antipodal cover of the line graph $L(G)$. If A has diameter at least four, then A is a line graph as well.*

The condition on the diameter of an antipodal cover is necessary, since, for example, the octahedron (i.e., $L(K_4) \cong K_{3(2)}$) has an antipodal double-cover which is not a line graph. Such an antipodal double-cover was constructed after Proposition 2.7.

The line graph operator is injective for the graphs with the minimum valency at least two. When we restrict to the line graphs, which are obtained from the graphs with the minimum valency at least two, L^{-1} is well defined and it can be applied it to a line graph which is an antipodal cover of a line graph. In this case we get a result similar to Theorem 3.1.

A path is called a *geodesic*, if it is a shortest path between its ends.

$$
\begin{tikzcd}
L(H) \arrow[d] \arrow[r, phantom, "\overset{L^{-1}}{\Longrightarrow}"] & H \arrow[d] \\
L(G) & G
\end{tikzcd}
$$

Fig. 3. The other direction.

Theorem 3.4. *Let G and H be two graphs with minimum valency at least two. If $L(H)$ is an antipodal r-cover with diameter $D \geqslant 4$ of $L(G)$, then H is an r-cover with diameter D of G. If the diameter of G is two or if any geodesic of length $D-1$ in $L(H)$ can be extended to a geodesic of length D, then H is an antipodal cover of G (see Fig. 3).*

The condition on the diameter of an antipodal cover is again necessary. Otherwise the line graph of the Petersen graph as distance-regular antipodal three-fold cover of K_5 would imply that the Petersen graph is an antipodal three-fold cover too, which is evidently not true.

4. Antipodal covers of $K_{v,v}$ and $K_v \times K_v$

Drake [6] has proved that a distance-regular antipodal cover with index r of $K_{v,v}$ is equivalent to a resolvable transversal design $\mathrm{TD}(v, v/r; r)$. In the extremal cases of r, Shad [12] and Delorme [5] have shown that distance-regular antipodal double-covers of $K_{v,v}$ are equivalent to Hadamard matrices, and Gardiner [7] has shown that distance-regular antipodal covers with maximal index, i.e., $r = v$, of $K_{v,v}$ are equivalent to affine planes of order v with a parallel class deleted. We will give a similar characterization of antipodal covers of the complete bipartite graphs $K_{v,v}$ and the lattice graphs $K_v \times K_v$.

A *'weak' resolvable transversal design* $\mathrm{WRTD}(v,r)$ is an incidence structure on rv points, partitioned into v groups of size r, and rv distinguished subsets, called lines, such that

(1) every line intersects each group in exactly one point,
(2) parallelism (i.e., being either equal or disjoint) is an equivalence relation on the lines,
(3) there are v parallel classes, each consisting of r lines (i.e., v *resolution classes*), and
(4) there exists a line through any two points if and only if they are from different groups.

Note that (4) implies $r \leqslant v$. If in a weak resolvable transversal design $\mathrm{WRTD}(v,r)$ a number of lines through any two points from different groups is constant, then this design is a resolvable transversal design $\mathrm{TD}(v, v/r; r)$. Any resolvable transversal design $\mathrm{TD}(v, v/r; r)$ satisfies the property (2). The dual of a resolvable transversal design $\mathrm{TD}(v, v/r; r)$ is actually a resolvable transversal design with the same parameters. We are now ready to characterize antipodal covers of the complete bipartite graphs and the lattice graphs.

Theorem 4.1. *There is a bijective correspondence between antipodal r-covers of $K_{v,v}$, antipodal r-covers of $K_v \times K_v$ and 'weak' resolvable transversal designs* WRTD(v,r). *In the extremal case, when $r = v$, the covers correspond to affine planes of order v with a parallel class deleted.*

Proof. By Propositions 2.7 and 2.4 the diameter of an antipodal cover must be in both cases equal to four. By Lemma 3.3, an antipodal cover of $K_v \times K_v$ is the line graph $L(H)$ of a graph H and, by Theorem 3.4, the graph H is an antipodal cover of $K_{v,v}$. Conversely, an antipodal cover H of $K_{v,v}$ is bipartite, thus by Theorem 3.1 the line graph $L(H)$ is an antipodal cover of $K_v \times K_v$. Therefore, antipodal covers of $K_{v,v}$ and $K_v \times K_v$ are equivalent.

Let $(\mathscr{P}, \mathscr{L})$ be the colour partition of H into 'points' and 'lines' and $\mathscr{D}$ the design with H as the incidence graph. Therefore, the fibres of H partition $\mathscr{P}$ into v groups of size r and $\mathscr{L}$ into v classes each consisting of r disjoint lines. Then diam$(H) > 2$ implies (1) and (3) for these classes. The definition of H implies that any two points (resp. lines) from different fibres have a common neighbour, i.e., lie on a line (resp. intersect in a point) therefore (4) and (2) are satisfied and $\mathscr{D}$ is a weak resolvable transversal design WRTD(v,r). If $r = v$ then any two lines are either parallel or they intersect in exactly one point, therefore the design is an affine plane with a parallel class deleted.

The proof that the incidence graph of a 'weak' resolvable transversal design $\mathscr{D}$ is an antipodal cover of $K_{v,v}$ is a straightforward verification of the definition. □

We close this section with a construction of a family of antipodal double-covers of $K_{v,v}$ which are not distance-regular. It is not hard to see that a non-distance-regular graph $K_2 \times C_6$ is the only antipodal double-cover of $K_{3,3}$. (Its line graph is the only antipodal double-cover of $K_3 \times K_3$.) We generalize this example. We start with two copies of the complete bipartite graph on $2v$ vertices with a matching deleted. Each copy is an antipodal double-cover with diameter three of K_v. Now, we connect each vertex of one copy with the vertex of the other copy which corresponds to its antipodal vertex and we get an antipodal double-cover of $K_{v,v}$. The existence of this family implies that antipodal double-covers of the complete bipartite graphs are not equivalent to Hadamard matrices although distance-regular antipodal double-covers are.

5. Conclusions

The requirement for a cover to be antipodal is restrictive enough that these graphs have a nice combinatorial structure. Our examples show that the structure gets even richer for larger diameter or larger covering index.

Lemma 2.1 and Corollary 3.2 seem to be reasonable tools for the study of the infinite families of feasible arrays of distance-regular antipodal covers of strongly regular graphs, however difficulties arise when there are not enough triangles outside maximal cliques, like in the point graph of a generalized quadrangle. The existence of distance-regular antipodal covers with diameter four of the Higman–Sims graph is the smallest open case.

There are cases in which the existence of antipodal distance-regular graph and its antipodal quotient are equivalent. For an illustrative example we need one more definition. A *generalized Odd graph* with diameter d (also called *regular thin near* $(2d+1)$-*gon*) is a distance-regular graph G with diameter d such that $a_1(G) = a_2(G) = \cdots = a_{d-1}(G) = 0$ and $a_d(G) > 0$.

Proposition 5.1. *If H is a bipartite antipodal distance-regular graph with odd diameter, then it is the bipartite double of its antipodal quotient G (i.e., $K_2 \otimes G$), which is a generalized Odd graph. Conversely, the bipartite double of a generalized Odd graph G is a bipartite distance-regular antipodal cover of G, with odd diameter.*

Proof. Let H have diameter $2d+1$ and index r. From the fact that $a_d(H) = a_{d+1}(H) = 0$, it follows that $r = 2$ (see [8] or [3, p. 142]). If we add $r = 2$ to the assumptions of the statement we get the known result from Brouwer et al. [3, Theorem 4.2.11]. The converse is also known, see Brouwer et al. [3, Theorem 1.11.1(vi)]. Therefore, we omit the remainder of the proof. □

The above result implies that there is a bijective correspondence between bipartite antipodal distance-regular graphs of odd diameter and their antipodal quotients. The known examples are the Desargues graph as the bipartite double of the Petersen graph, the five-cube, the double Hoffman–Singleton, the double Gewirtz, the double 77-graph (i.e., the bipartite double of the unique strongly regular graph with intersection array $\{21, 20; 1, 4\}$, and the double Higman–Sims, see [3].

If a distance-regular antipodal cover H with diameter four of a strongly regular graph G with intersection array $\{k, k-a_1-1; 1, c_2\}$ is bipartite, then $a_2(H) = 0$ implies $k = c_2$. It means that G is the complete multipartite graph with t classes of size m, i.e., $K_{t(m)}$, and $a_1(H) = 0$ implies $t = 2$, i.e., G is the complete bipartite graph $K_{k,k}$.

Perhaps the correspondence between the antipodal covers of the point and the line graphs of the same incidence structure can be extended to other graphs derived from incidence structures.

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